areas in Fig. 1 represent the domains of optimal stabilization by forces with partial dissipation corresponding to Eqs. (4.10)–(4.12) and coincident with the domains of fulfillment of the sufficient conditions of stability with the exception of the straight line $a = 2$ for solution (4.10), the straight line $a = 0$ for solution (4.11), and the straight line $b = -1$ for solution (4.12) on which the conditions of optimal stabilization are not fulfilled in the first approximation. However, consideration of the nonlinear terms in the equations of perturbed motion indicate that the conditions of optimal stabilization are also fulfilled on these straight lines.

BIBLIOGRAPHY

1. Krasovskii, N. N., Problems of stabilization of controlled motions. In: Malkin, I. G., Theory of Stability of Motion, Appendix 4. Moscow, "Nauka", 1966.
2. Letov, A. M., Flight Dynamics and Control. Moscow, "Nauka", 1969.
3. Rumiantsev, V. V., On the stability of motion with respect to some of the variables. Vestnik Mosk. Gos. Univ., №4, 1957.
4. Krasovskii, N. N., A generalization of the theorems of Liapunov's second method. In: Malkin, I. G., Theory of Stability of Motion, Appendix 3, 2nd ed. Moscow, "Nauka", 1966.
5. Chetaev, N. G., Stability of Motion, 3rd ed. Moscow, "Nauka", 1965.
6. Krasovskii, N. N., On a certain property of the gyroscopic stabilizability of a controllable conservative mechanical system. Izv. Akad. Nauk SSSR, Technicheskaia Kibernetika №5, 1964.
7. Gabrielian, M. S. and Krasovskii, N. N., On the problem of stabilization of a mechanical system. PMM Vol. 28, №5, 1964.
8. Pozharitskii, G. K., On asymptotic stability of equilibria and stationary motions of mechanical systems with partial dissipation. PMM Vol. 25, №4, 1961.
9. Rumiantsev, V. V., On the Stability of Steady Motions of Satellites. Moscow, Izd. VTs Akad. Nauk SSSR, 1967.
10. Corduneanu, C., Sur la stabilité partielle. Revue Roumaine de Mathématiques pure et appliquée. Vol. 9, №3, 1964.

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ON THE PRECISION OF OPTIMAL CONTROL OF THE FINAL STATE

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The Bellman partial differential equation involved in the synthesis of a stochastically optimal control of the final state of a linear system is considered. Approximate formulas and estimates of the solution are derived on the basis of the solution of the Bellman equation for the determinate variant of the problem. A numerical method of solution is proposed. The problem in the one-dimensional case is reduced to an integral equation of the first kind; a finite formula for the solution is derived under certain additional assumptions.

1. The initial equations. One of the problems investigated in [1] is that of synthesizing the bounded stochastically optimal control of the final state of a linear system described by the equations (*)

$$x^{\bullet} = Q(t)u + \xi(t) \qquad (0 < t \leqslant T, \quad u(t) \in U(t)) \tag{1.1}$$

Here t is the time; x, u, $\xi(t)$ are the coordinate, control, and random disturbance vectors of orders n, m, n, respectively; $Q(t)$ is the given matrix of coefficients, $U(t)$ is the closed set of permissible values of the control vector, and $[0, T]$ is the given time integral. Information on the present state of the system is provided by the vector $y(t) = G(t)x(t) + h(t)$, where $G(t)$ is a given matrix and $h(t)$ is the vector of random errors involved in obtaining the information. It was assumed that the forces ξ and h were white noises and that they shared a normal distribution law with the vector $x(0)$. The object was to find the optimal control operator $u(t) = u\{y(\tau), u(\tau);\ 0 \leqslant \tau < t\}$ which minimizes the precision estimate $S = M\omega[x(T)]$, i.e. the mathematical expectation of the prescribed scalar function $\omega[x(T)]$ of the vector defining the final state of the system.

As is shown in [1], the optimal control can be expressed in the form of a function $u(t, z(t))$, where $z(t) = M[x(t) \diagdown y(\tau), u(\tau); 0 \leqslant \tau < t]$ is the nominal mathematical expectation of the vector $x(t)$ for a known realization of $y(\tau)$, $u(\tau)$ in the interval $[0, t)$. The function $u(t, z)$ must be determined together with the corresponding a posteriori estimate $S(t, z(t)) = M\{\omega[x(T)] \diagdown y(\tau), u(\tau);\ 0 \leqslant \tau < t\}$ from the nonlinear Bellman second-order partial differential equation

$$-S_t = \min_u (S_z, Q(t)u) + {}^1/_2 \operatorname{sp}(S_{zz}R(t)) \qquad (0 \leqslant t < T) \tag{1.2}$$

with the following boundary condition at $t = T$:

$$S(T, z) = \Psi(z) = \int \omega(z + x)\, q(T, x)\, dx \tag{1.3}$$

$$q(T, x) = [(2\pi)^n |C(T)|]^{-1/2} \exp {}^1/_2 (-C^{-1}(T)x, x)$$

Here z is an n-dimensional vector; S_t is the partial derivative of the Bellman function $S(t, z)$ with respect to t; S_z and S_{zz} are the vector of the first partial derivatives and the matrix of the second partial derivatives of the function $S(t, z)$ with respect to the components of the vector z; $(.,.)$ denotes the scalar multiplication of vectors, sp is the trace of the matrix, $R(t)$ and $C(T)$ are bounded nonnegative-definite square matrices; $|C(T)|$ is the determinant of the matrix $C(T)$; $q(T, x)$ is the density of the normal distribution of the random vector with the correlation matrix $C(T)$. Integration in formula (1.3) is over the entire space of the vector integration variable x (this statement will hold for all cases where the integration limits are not indicated.

The matrices $R(t)$ and $C(T)$ are defined by the formulas

$$R(t) = B^{\bullet}(T, t), \quad B(T, t) =$$

$$= M[(x(T) - z(t))(x(T) - z(t))' \diagdown u(\tau) = 0,\ \tau > t] \tag{1.4}$$

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*) We note that the more general system

$$x_1^{\bullet} = A(t)x_1 + Q_1(t)u + \xi_1(t)$$

is reducible to the form (1.1) by the substitution of variables $x = L(T)L^{-1}(t)x_1$, where $L(t)$ is the fundamental matrix of the homogeneous system.

$$C(T) = B(T, T) = M[(x(T) - z(T))(x(T) - z(T))'] \quad \text{(cont.)}$$

where the dot denotes differentiation with respect to the time t and the prime denotes transposition. They can be expressed in terms of the system coefficients and the characteristics of the laws of distribution of the processes $\xi(t)$ and $h(t)$ and of the vector $x(0)$. If $x(0) - Mx(0) = \xi(t) - M\xi(t) \equiv 0$, then $C(T) = R(t) \equiv 0$. In the absence of errors in receiving the information the matrix $R(t)$ coincides with the intensity of the white noise $\xi(t)$ in [2].

A theorem on the existence and uniqueness of the solution of equations of the (1.2), (1.3) type is proved in [3]. In the present paper we shall be concerned with obtaining finite formulas for the approximate and exact solutions of Cauchy problem (1.2), (1.3) suitable for practical use. We shall also consider in more detail the case where the set $U(t)$ takes the form of the parallelepiped

$$|u^{(i)}(t)| \leqslant l^{(i)}(t) \qquad (i = 1, \ldots, m) \tag{1.5}$$

bounding the components $u^{(i)}(t)$ of the control vector. Minimization with respect to the parameter u in Eq. (1.2) in this case can be effected in explicit form. If the system is described by the scalar equation

$$x^{\cdot} = u + \xi \qquad (0 < t \leqslant T) \tag{1.6}$$

and if the set $U(t)$ is the segment $|u(t)| \leqslant l(t)$ with the possible deletion of some interior intervals, then the Bellman equations become

$$-S_t = -l(t)|S_x| + {}^1/_2 R(t) S_{xx} \qquad (0 \leqslant t < T) \tag{1.7}$$

$$S(T, x) = \Psi(x) \tag{1.8}$$

(where the symbol for the phase coordinate is different from that of (1.2), (1.3): the alteration will prove useful below).

We note that a system in which $n = 1$, $m \neq 1$ can be reduced to the form (1.6) by introducing the scalar controlling parameter $u_1 = Q(t)u - u_0(t)$, where $2u_0(t) = \max_u Q(t)u + \min_u Q(t)u$ and where the new perturbation $\xi_1(t) = \xi(t) + u_0(t)$. Here

$$2l(t) = \max_u Q(t)u - \min_u Q(t)u$$

2. Solution of the Bellman equation for a system with complete information. The case with complete information, where the initial conditions and perturbations of the system are given, can be considered as a special case of a system with incomplete information by setting the initial deviation $x(0) - Mx(0)$ and perturbation $\xi(t)$ identically equal to zero (*). The matrices $R(t)$ and $C(t)$ in this case are equal to zero and Eqs. (1.1)–(1.3) become

$$x^{\cdot} = Q(t)u \tag{2.1}$$

$$-S_t^\circ = \min_u (S_x^\circ, Q(t)u) \qquad (0 \leqslant t < T, u \in U(t)) \tag{2.2}$$

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*) A system such that $\xi(t) = M$, $\xi(t) \neq 0$ is reducible to a system without perturbations by converting to the new coordinates

$$x_1 = x + \int_t^T \xi(\tau)\, d\tau$$

$$S^\circ(T,\ x) = \omega(x) \tag{2.3}$$

In order to distinguish the determinate variant of the problem from the stochastic one, we mark the a posteriori estimate in the determinate problem with a zero superscript; in other words, we denote it by the symbol $S^\circ(t,\ x)$. Let us consider some important particular cases where the a posteriory estimate $S^\circ(t,\ x)$ can be found from the equations of motion (2.1).

1°. Let the coordinate and control vectors be one-dimensional. The Bellman equations in this case can be written as

$$-S_t^\circ = -l(t)\,|S_x^\circ|, \qquad S^\circ(T, x) = \omega(x) \tag{2.4}$$

Let us suppose that the function $\omega(x)$ is piecewise-continuous, that it has a single minimum at some point x°, and that it is nondecreasing for $x \geqslant x^\circ$ and nonincreasing for $x \leqslant x^\circ$. The definition of the a posteriori estimate

$$S^\circ(t,\ x) = \min_u \omega\,[x(T) \setminus x(t) = x] \tag{2.5}$$

and Eqs. (2.1) imply that the function $S^\circ(t, x)$ in this case is given by the equations

$$\begin{aligned} &S^\circ(t, x) = \omega\,[x - b(t,\ T)\operatorname{sgn}(x - x^\circ)] \qquad (|x - x^\circ| \geqslant b(t, T)) \\ &S^\circ(t, x) = \omega\,[x^\circ] \qquad (|x - x^\circ| \leqslant b(t, T)), \quad b(t,\ T) = \int_t^T l(\tau)\,d\tau \end{aligned} \tag{2.6}$$

Substituting function (2.6) into Eqs. (2.4), we can verify the fact that this function is the solution of Cauchy problem (2.4) in the ordinary sense [4] provided that $\omega(x)$ is also continuously differentiable.

2°. Let the function $\omega(x)$ be defined by the equations

$$\omega(x) = 1 \quad (x \notin D), \qquad \omega(x) = 0 \quad (x \in D) \tag{2.7}$$

where D is a given closed domain in the n-dimensional vector space. This choice of the function $\omega(x)$ optimizes the entry of the vector $x(T)$ into the given domain D. We define the domain $D(t,\ T)$ as the domain of those values of the vector x at the instant t from which system (2.1) can be brought into the domain D at the instant T. The control $u(\tau,\ x)$, $t \leqslant \tau \leqslant T$ is optimal if and only if this control ensures the transfer of the system at the instant T into the domain D for all $x(t) \in D(t,\ T)$. This implies that the a posteriori estimate corresponding to the optimal control is given by the equations

$$S^\circ(t, x) = 1 \quad (x \notin D(t, T)), \qquad S^\circ(t, x) = 0 \quad (x \in D(t, T)) \tag{2.8}$$

Because it is discontinuous, function (2.8) cannot be the solution of Cauchy problem (2.2), (2.3) in the ordinary sense. In the one-dimensional case it can be regarded as the solution in the following generalized sense: the sequence of solutions (2.6) associated with the sequence $\omega_1(x)$, $\omega_2(x)$, ... of differential uniminimal functions which converges to function (2.7), converges to function (2.8). (This statement probably continues to hold in the multidimensional case). In contrast to the usual definition of the generalized solution [4], the above definition does not require that the convergence be uniform.

The domain $D(t,\ T)$ is defined by the inequality $T^\circ(t,\ x) \leqslant T - t$, where $T^\circ(t,x)$ is the minimum time to bring the system into the domain D from the position x at the instant t. Construction of the domain $D(t,\ T)$, which involves solution of the time-optimal operation problem [2], is quite difficult in the multidimensional case. Let the set $U(t)$ and the domain D be parallelepipeds defined by inequalities (1.5) and

$$|x^{(i)} - a_0^{(i)}| \leqslant d_0^{(i)} \qquad (i = 1, \ldots, n) \tag{2.9}$$

where $a_0^{(i)}$, $d_0^{(i)}$ are given quantities. In this case it is a relatively simple matter to determine the rectangular domain

$$|x^{(i)} - a_0^{(i)}| \leqslant d^{(i)}(t) \qquad (i = 1, \ldots, n)$$

circumscribing the domain $D\ (t,\ T)$. We denote the former by $D^*\ (t, T)$. The dimendions $d^{(i)}\ (t)\ (i = 1, \ldots, n)$ are given by the formula

$$d^{(i)}(t) = d_0^{(i)} + \max_u \{x^{(j)}(T) \setminus x^{(i)}(t) = 0\} = d_0^{(i)} + \sum_{j=1}^{m} \int_t^T |Q_{(j)}^{(i)}(\tau)|\, l^{(j)}(\tau)\, d\tau \tag{2.10}$$

The function defined by the equations

$$S^*(t, x) = 1 \quad (x \notin D^*(t, T)), \qquad S^*(t, x) = 0 \quad (x \in D^*(t, T)) \tag{2.11}$$

serves as the lower estimate for a posteriori estimate (2.8).

In the case of arbitrary D and $U\ (t)$ they can always be contained in the paralelepipeds D_1 and $U_1\ (t)$; the corresponding domain $D_1^*\ (t,\ T)$ can be determined from formulas (2.10). This domain contains the domain $D^*\ (t,\ T)$. The function $S_1^*\ (t, x)$ defined by Eqs. (2.11), where the domain $D^*\ (t,\ T)$ is replaced by the domain $D_1^*\ (t,\ T)$, is a lower estimate for the functions $S^*\ (t, x)$ and $S^\circ\ (t, x)$,

$$S_1^*(t, x) \leqslant S^*(t, x) \leqslant S^\circ(t, x) \tag{2.12}$$

3°. Let the function $\omega\ (x)$ attain its minimum at some point x°. In this case the a posteriori estimate $S^\circ\ (t,\ x)$ satisfies the condition

$$S^\circ(t,\ x) = \omega(x^\circ) \quad (x \in D_{x^\circ}(t, T)) \tag{2.13}$$

Here $D_{x^\circ}\ (t,\ T)$ is the domain of values of the coordinates x at the instant t from which the system can be brought to the point x° at the instant T. Equation (2.13) follows from the definition (2.5) of a posteriori estimate.

3. Estimates and approximate formulas for the solution of the Bellman equation for a stochastic system. In this section we consider several variants of the approximate solution of Cauchy problem (1.2), (1.3), each of which is a lower estimate of the exact solution. Let us consider the function defined by the formula

$$F_1(t, z) = \int S_1(t, x)\, P(t, T, x - z)\, dx = \int S_1(t, x + z)\, P(t, T, x)\, dx \tag{3.1}$$

Here $S_1\ (t,\ x)$ is the solution of Eq. (2.2) under condition (1.3); $P\ (t,\ T, x - z)$ is the fundamental solution [5] of the equation

$$-P_t = {}^1/_2\, \mathrm{sp}\,(P_{zz} R(t)) \tag{3.2}$$

corresponding to system (1.1) in the absence of control. The function $P\ (t,\ T,\ x)$ is given by the formulas [6]

$$P(t, T, x) = (2\pi)^{-n/2} |K_1(t, T)|^{-1/2} \exp(-{}^1/_2\, K_1^{-1}(t, T)\, x, x) \tag{3.3}$$

$$K_1(t, T) = \int_t^T R(\tau)\, d\tau = B(T, t) - C(T) \tag{3.4}$$

It is equal to the density of the normal distribution of the vector $z\ (T) - z\ (t)$ under the assumption that $u\ (\tau) = 0$ for $\tau \geqslant t$. This follows from Eq. (1.4) and from the uncorrelatedness of the vectors $x\ (T) - z\ (T)$ and $z\ (T) - z\ (t)$. The function

$S_1(t, x)$ is the a posteriori estimate for optimal control of the system with complete information under the assumption that the optimizable final-state function is given by formula (1.3). By construction, the function $F_1(t, z(t))$ is equal to the a posteriori estimate for the case of optimal control of a system which receives at the instant t all the additional information $y(\tau)$ $(t \leqslant \tau < T)$ in addition to the information $y(\tau)$, $u(\tau)$ $(0 \leqslant \tau < t)$ available at that instant. With optimal control the additional information cannot decrease the control precision (increase the precision estimate), so that function $F_1(t, z)$ is a lower estimate of the Bellman function,

$$F_1(t, z) \leqslant S(t, z) \tag{3.5}$$

The difference $F_1(t, z) - S(t, z)$ vanishes in the two following cases:

1) if the matrix $R(t)$ is equal to zero for all t (this corresponds to the absence of measurements or to the absence of perturbations and initial deviations);

2) if the system is not controlled (i.e. if the set $U(t)$ consists of the single point $u = 0$).

The function $F_1(t, z)$ can therefore be considered as an approximate solution of Cauchy problem (1.2), (1.3), provided the matrix $R(t)$ is small or the dimensions of the set $U(t)$ are small.

Let the set $U(t)$ be a closed domain and let system (2.1) be controlled [7] in the interval $[T - \delta, T]$ for any arbitrarily small $\delta > 0$. The difference between the exact and approximate solutions then also tends to zero in the case of infinite expansion of the domain $U(t)$,

$$F_1(t, T)\big|_{r(U)=\infty} = \lim_{r(U)\to\infty} F_1(t, z) = S(t, z)\big|_{r(U)=\infty} \tag{3.6}$$

Here $r(U)$ is the radius of a sphere with its center at the point $u = 0$ which fits completely in the domain $U(t)$ for all t from the interval $[T - \delta, T]$ for some $\delta > 0$.

In fact, if the control is unbounded, it is optimal if and only if $z(T) = z^\circ$, where z° is the minimum point of the function $\Psi(z)$ in Eq. (1.3). The a priori and a posteriori estimates S and $S(t, z)$ then attain their minimum values,

$$\min_u S\big|_{r(U)\to\infty} = \lim_{r(U)\to\infty} S(t, z) = \Psi(z^\circ) \qquad (0 \leqslant t < T) \tag{3.7}$$

On the other hand, formulas (3.1), (3.3) and (2.13) with $\omega(x)$ replaced by $\Psi(x)$ as well as the fact that the domain $D_{z^\circ}(t, T)$ expands without limit with expansion of U enable us readily to prove the equation

$$F_1(t, z)\big|_{r(U)\to\infty} = \lim_{r(U)\to\infty} F_1(t, z) = \Psi(z^\circ) \qquad (0 \leqslant t < T) \tag{3.8}$$

Comparison of formulas (3.7) and (3.8) yields the required result.

Let us consider the function $F_2(t, z)$ defined by formulas (3.1), (3.3), where the function $S_1(t, x)$ is replaced by $S^\circ(t, x)$ (Eqs. (2.2), (2.3)) and the matrix $K_1(t, T)$ by the matrix

$$K_2(t, T) = C(T) + \int_t^T R(\tau)\, d\tau = B(T, t) \tag{3.9}$$

The function $F_2(t, z)$ is a lower estimate for $F_1(t, z)$.

In fact, let the system receive the information $y_1 = x(T) \setminus u(\tau) = 0\ (\tau > t)$ in the sense of the criterion $S = M\omega[x(T)]$ at the instant t. The function defined by Eqs. (3.3), (3.9), (1.4) is equal to the normal distribution density of the vector $y_1 - z(t)$. This means that the function $F_2(t, z(t))$ is an a posteriori estimate for the given hypothetical system with optimal control. The function $F_2(t, z)$ is therefore a lower estimate for the functions $F_1(t, z)$ and $S(t, z)$.

Exactly as in the case of the function $F_1(t, z)$ we can show that the difference $S(t, z) - F_2(t, z)$ tends to zero as the dimensions of the set $U(t)$ or the dispersions of the random forces go to zero.

In the case of (2.7) the functions

$$F^*(t, z) = \int S^*(t, x) P(t, T, z - x)\, dx, \quad F_1^*(t, z) = \\ = \int S_1^*(t, x) P(t, T, z - x)\, dx \tag{3.10}$$

are clearly lower estimates for $F_2(t, z)$.

Here the function $P(t, T, x)$ is specified in the same way as in the determination of $F_2(t, z)$ (formulas (3.3), (3.9)). The functions $S^*(t, x)$ and $S_1^*(t, x)$ can be found from formula (2.11); they are lower estimates (2.12) for $S^\circ(t, x)$. This implies the validity of the chain of inequalities

$$F_1^*(t, z)) \leqslant F^*(t, z) \leqslant F_2(t, z) \leqslant F_1(t, z) \leqslant S(t, z)$$

The estimates $F_1^*(t, z)$, $F^*(t, z)$, $F_2(t, z)$ are coarser than $F_1(t, z)$ but are easier to compute. In particular, the function $F_1^*(t, z)$ is given by finite formulas.

4. Numerical solution of the Bellman equation. Let the function $\omega(x)$ be given by Eqs. (2.7). In this case the a posteriori estimate $S(t, z(t))$ is the conditional probability that the point $x(T)$ will lie outside the domain D under optimal control. This means that the function $S(t, z)$ satisfies the inequalities

$$F_1^*(t, z) \leqslant S(t, z) \leqslant 1 \tag{4.1}$$

Let us define the rectangular domain $\Omega(\varepsilon, t)$ in the coordinates z by means of the formulas

$$|z^{(i)} - a_0^{(i)}| \leqslant d_2^{(i)}(\varepsilon, t) \qquad (i = 1, \ldots, n)$$

$$d_2^{(i)}(\varepsilon, t) = d_1^{(i)}(t) + \gamma(\varepsilon) \sqrt{[K_2(t, T)]_{(i)}^{(i)}} \tag{4.2}$$

Here $\gamma(\varepsilon)$ is the inverse of the function

$$^1/_2 - \Phi(\gamma) = {}^1/_2 - (2\pi)^{-1/2} \int_0^\gamma e^{-\frac{x^2}{2}}\, dx \tag{4.3}$$

ε is a given sufficiently small positive number, and $d_1^{(i)}(t)$ $(i = 1, \ldots, n)$ are the dimensions of the domain $D_1^*(t, T)$ defined by formulas (2.10); $[K_2(t, T)]_{(i)}^{(i)}$ is the corresponding diagonal element of matrix (3.9). From Eqs. (2.11) and (3.10) we see that the function $F_1^*(t, z)$ nowhere exceeds unity and that it differs from unity by an amount smaller than ε outside $\Omega(\varepsilon, t)$ (the dimensions of the domain $\Omega(\varepsilon, t)$ were determined precisely from this condition). Together with inequalities (4.1) this implies that the function $S(t, z)$ for $z \notin \Omega(\varepsilon, t)$ also differs from unity by an amount smaller than ε. This fact enables us to propose the following numerical method for solving Eqs. (1.2), (1.3).

A sufficiently small number ε is chosen and the solution is set equal to unity at the boundary of the domain $\Omega(\varepsilon, t)$. As a result the initial Cauchy problem (1.2), (1.3) is replaced by the first boundary value problem [5] for Eq. (1.2) under conditions (1.3) and

$$S(t, z)\big|_{z \in \Gamma(\varepsilon, t)} = 1 \tag{4.4}$$

Here $\Gamma(\varepsilon, t)$ is the boundary of the domain $\Omega(\varepsilon, t)$. Because the solution of Eq. (1.2) is continuously dependent on the boundary conditions [3] the error due to this substitution is on the order of ε. The domain $\Omega(\varepsilon, t)$ in Eq. (4.4) can be replaced by the broader domain $\Omega(\varepsilon, 0)$ (see formulas (4.2), (3.9), (2.10)). In this case the problem can be solved numerically using a difference scheme [8] with a node net independent of the number k of the layer $t_k = T - k\Delta$, $k = 0, 1, \ldots, N$. The solving procedure is even simpler in case (1.5).

The proposed method is clearly applicable in the case of an arbitrary function $\omega(x)$ provided it has a finite limit $\omega(\infty)$ as $|x| \to \infty$. The given domain D (or the domain circumscribed about D) in the formulas defining the dimensions of the domain $\Omega(\varepsilon, t)$ is replaced in this case by the dimensions of the rectangular domain beyond whose limits the function $\omega(x)$ differs from the limiting value $\omega(\infty)$ by an amount smaller than ε. The boundary value of the function $S(t, x)$ in Eq. (4.4) is assumed to be equal to $\omega(\infty)$.

5. A finite formula for the solution of the Bellman equation in the one-dimensional case. Let us consider Eqs. (1.7), (1.8) corresponding to one-dimensional system (1.6). We assume that the coefficients $l(t)$ and $R(t)$ are continuous in t and positive in the interval $[0, T]$

$$l(t) > 0, \qquad R(t) > 0 \qquad (0 \leqslant t \leqslant T) \tag{5.1}$$

and that the function $\Psi(x)$ is even, nondecreasing for $x \geqslant 0$, continuously differentiable and bounded,

$$\Psi(-x) = \Psi(x), \quad \Psi'(x) \geqslant 0 \qquad (x \geqslant 0) \tag{5.2}$$

The primes here and below denote derivatives with respect to x. Fulfillment of the conditions of existence and uniqueness of the solution of Cauchy problem (1.7), (1.8) is guaranteed in this case [3]. Let us show that the solution is given by the equations

$$S(t, x) = s(t, x) \quad (x \geqslant 0) \qquad S(t, x) = s(t, -x) \quad (x \leqslant 0) \tag{5.3}$$

Here $s(t, x)$ is the solution of the linear parabolic equation

$$-s_t = -l(t)s_x + {}^1\!/_2 R(t)s_{xx} \quad (x > 0, \ 0 \leqslant t < T) \tag{5.4}$$

satisfying the conditions

$$s(T, x) = \Psi(x) \qquad (x \geqslant 0) \tag{5.5}$$

$$s_x(t, x)|_{x=0} = 0 \quad (0 \leqslant t \leqslant T) \tag{5.6}$$

(This equation is obtainable from Eq. (1.7) by replacing $|S_x|$ by s_x.)

P r o o f. Let $S(t, x)$ be the solution of Eqs. (1.7), (1.8). Then the function $S(t, -x)$ is also a solution. This can be verified simply by substituting this function into the indicated equations. Because of the uniqueness of the solution this implies that the function $S(t, x)$ is even in x, i.e. that $S(t, -x) = S(t, x)$. The latter equation and the continuous differentiability of $S(t, x)$ with respect to x (this requiremet occurs in the definition of the solution [4]) imply that $S_x(t, x)|_{x=0} = 0$ for all t from the interval $[0, T]$. Bearing in mind the fact that $S_x(T, x) = \Psi'(x)$ is nonnegative for $x \geqslant 0$, we can assume that $S_x(t, x) \geqslant 0$ for all t from the interval $[0, T]$ and $x \geqslant 0$. In this connection let us consider the boundary value problem for linear parabolic equation (5.4) under conditions (5.5), (5.6). This boundary value problem has a unique solution [5]. Differentiating Eqs. (5.4), (5.5), we conclude that the derivative $s_x(t, x)$ satisfies the same condition (5.4) and the conditions

$$s_x(T, x) = \Psi'(x) \geqslant 0 \quad (x \geqslant 0), \quad s_x(t, x)|_{x=0} = 0$$

On the basis of the maximum principle for a linear parabolic equation [5] this implies that $s_x(t, x) \geqslant 0$ for $x \geqslant 0$. Thus, $s_x(t, x) = |s_x(t, x)|$ for $x \geqslant 0$, so that the function $s(t, x)$ satisfies Eqs. (1.7), (1.8) for $x > 0$. The function defined by Eqs. (5.3) is the solution of Cauchy problem (1.7), (1.8), since it satisfies condition (1.8) for all x and Eq. (1.7) for $x \neq 0$; the continuity of this function and its derivatives S_x, S_{xx}, S_t for all x and t means that it also satisfies Eq. (1.7) for $x = 0$.

Initial problem (1.7), (1.8) thus reduces to the solution of the second boundary value problem [5] for linear equation (5.4) under conditions (5.5), (5.6). Let us show that this problem can in turn be reduced to the solution of some integral equation of the first kind.

Let us continue the function $s(T, x)$ in Eq. (5.5) to negative values of x in some way which preserves continuity,

$$s(T, x) = \Psi(x) \quad (x \geqslant 0), \quad s(T, x) = \Psi_1(-x) \quad (x \leqslant 0) \tag{5.7}$$

Here $\Psi_1(x)$ is a temporarily unknown differentiable function which satisfies the condition

$$\Psi_1(0) = \Psi(0) \tag{5.8}$$

We seek the solution of the boundary value problem (5.4)–(5.6) in the form of the solution of Cauchy problem (5.4), (5.7) in the strip $-\infty \leqslant x < \infty$, $0 \leqslant t < T$ using the familiar formula [5]

$$s(t, x) = \int_{-\infty}^{+\infty} p(t, T, x - y) \quad s(T, y)\, dy = \int_0^{\infty} [\Psi(y)\, p(t, T, x - y) + \\ + \Psi_1(y)\, p(t, T, x + y)]\, dy \tag{5.9}$$

Here $p(t, T, x - y)$ is the fundamental solution of Eq. (5.4) determined from the formulas

$$p(t, \tau, x - y) = [2\pi t_1(t, \tau)]^{-1/2} \exp \frac{-[x - b(t, \tau) - y]^2}{2t_1(t, \tau)} \tag{5.10}$$

$$t_1(t, \tau) = \int_t^{\tau} R(s)\, ds, \quad b(t, \tau) = \int_t^{\tau} l(s)\, ds \tag{5.11}$$

Formula (5.10) can be obtained as follows. We use the substitution of variables

$$t_1 = t_1(t, T) \quad x_1 = x - b(t, T), \quad s_1(t_1, x_1) = s[t(t_1), x(t_1, x_1)]$$

to reduce Eq. (5.4) to an ordinary heat conduction equation whose fundamental solution is given by the familiar formula

$$p_1(t_1 - \tau_1, x_1 - y_1) = [2\pi(t_1 - \tau_1)]^{-1/2} \exp \frac{-[x_1 - y_1]^2}{2(t_1 - \tau_1)}$$

We note that the inverse transform $t(t_1)$, $x(t_1, x_1)$ exists and is unique by virtue of our assumption of positive $R(t)$ and $l(t)$. Reconverting to our original variables, we obtain formula (5.10).

Differentiating Eq. (5.9) with respect to x and applying condition (5.6), we obtain an equation of the first kind [9] in the derivative $\Psi_1'(x)$,

$$\int_0^{\infty} \left[\Psi'(x) \exp\left(-\frac{2b(t, T)}{t_1(t, T)} x\right) - \Psi_1'(x)\right] \exp \frac{-[x^2 - 2b(t, T)x]}{2t_1(t, T)}\, dx = 0 \quad (0 \leqslant t \leqslant T) \tag{5.12}$$

As a result of Eq. (5.8) the function $\Psi_1(x)$ is given by the formula

$$\Psi_1(x) = \Psi(0) + \int_0^x \Psi_1'(y)\,dy \qquad (x \geqslant 0) \tag{5.13}$$

In those cases where Eq. (5.12) has a sufficiently smooth solution, the solution of Cauchy problem (1.7), (1.8) can be found from formulas (5.3), (5.9), (5.13).

Integrating the left side of Eq. (5.12) by parts, we obtain an integral equation for determining the function $\Psi_1(x)$ directly,

$$\int_0^\infty K(x, t)\,\Psi_1(x)\,dx = f(t) \qquad (0 \leqslant t \leqslant T)$$

The kernel $K(x, t)$ and the absolute term $f(t)$ of the integral equation are given by the formulas

$$K(x, t) = \frac{x - b(t, T)}{t_1(t, T)} \exp\left(-\frac{x^2 - 2b(t, T)\,x}{2t_1(t, T)}\right)$$

$$f(t) = \int_0^\infty \Psi(x)\,\frac{x + b(t, T)}{t_1(t, T)} \exp\left(-\frac{x^2 + 2b(t, T)\,x}{2t_1(t, T)}\right) dx$$

1°. Let the ratio of coefficients of Bellman equation (1.7) be equal to a constant,

$$\frac{l(t)}{R(t)} = \alpha(t) = \alpha = \text{const} \tag{5.14}$$

In this case the solution of integral equation (5.12) is given by the finite formula

$$\Psi_1'(x) = e^{-2\alpha x}\,\Psi'(x) \tag{5.15}$$

In fact, as we see from Eqs. (5.11), in case (5.14) we have $b(t, T)\,t_1^{-1}(t, T) = \alpha$, so that Eq. (5.12) becomes an identity when we make substitution (5.15). Substituting expression (5.15) into formula (5.13), we obtain

$$\Psi_1(x) = e^{-2\alpha x}\Psi(x) + 2\alpha \int_0^x e^{-2\alpha y}\Psi(y)\,dy \tag{5.16}$$

With allowance for the above equation formula (5.9) becomes

$$s(t, x) = \int_0^\infty g_0(t, x; T, y)\,s(T, y)\,dy, \qquad s(T, y) = \Psi(y) \tag{5.17}$$

$$g_0(t, x; T, y) = p(t, T, x - y) + e^{-2\alpha y} p(t, T, x + y) - \\ - 2\alpha e^{-2\alpha y}\left[\Phi\left(\frac{x + y - b(t, T)}{\sqrt{t_1(t, T)}}\right) - {}^1/_2\right] \tag{5.18}$$

Here $\Phi(x)$ is probability integral (4.3).

Formulas (5.17), (5.18) can be derived as follows. Substituting expression (5.16) for the function $\Psi_1(x)$ into formula (5.9), we obtain

$$s(t, x) = \int_0^\infty [p(t, T, x - y) + e^{-2\alpha y} p(t, T, x + y)]\,\Psi(y)\,dy + \\ + 2\alpha \int_0^\infty p(t, T, x + y) \int_0^y e^{-2\alpha z}\Psi(z)\,dz\,dy \tag{5.19}$$

Making use of the identity

$$p(t, T, x+y) = \frac{d}{dy}\left[\Phi\left(\frac{x+y-b(t, T)}{\sqrt{t_1(t, T)}}\right) - {}^1/_2\right]$$

we obtain

$$\int_0^\infty p(t, T, x+y)\int_0^y e^{-2\alpha z}\Psi(z)\,dz\,dy = \left[\Phi\left(\frac{x+y-b(t, T)}{\sqrt{t_1(t, T)}}\right) - {}^1/_2\right]\int_0^y e^{-2\alpha z}\Psi(z)\,dz\,\Big|_{y=0}^{y=\infty} -$$

$$-\int_0^\infty \left[\Phi\left(\frac{x+y-b(t, T)}{\sqrt{t_1(t, T)}}\right) - {}^1/_2\right] e^{-2\alpha y}\Psi(y)\,dy$$

On substitution of limits the first term vanishes, as a result of which the formula (5.19) with allowance for notation (5.18) assumes the form (5.17).

In some cases it is more convenient to determine the function $g_0(t, x; T, y)$ from the formula

$$g_0(t, x, T, y) = -\frac{\partial}{\partial y}\int_\infty^x [p(t, T, z-y) - e^{-2\alpha y}p(t, T, z+y)]\,dz \tag{5.20}$$

which yields an expression identically equal to (5.18).

Although formulas (5.17),(5.18) were obtained under assumption of continuous differentiability and boundedness of the function $\Psi(x)$, they are also applicable if the function $\Psi(x)$ has discontinuities of the first kind, as well as if it is unbounded at infinity provided it does not increase more rapidly than the function

$$\exp\left[{}^1/_2\left(\int_0^T R(\tau)\,d\tau - \varepsilon\right)^{-1} x^2\right]$$

where $\varepsilon > 0$. In fact, in these cases function (5.17),(5.18) remains sufficiently smooth and satisfies (by construction) Eq. (5.4) and conditions (5.5),(5.6). The above formulas for the exact solution can be used to analyze the exactness of various approximate methods of solving the Bellman equation by comparing the approximate and exact solutions for $\alpha(t) = \text{const}$.

2°. If $\alpha(t) = l(t)/R(t)$ is a piecewise-constant function which assumes the values $\alpha_1, \ldots, \alpha_N$ in the intervals $[0, t_1], (t_1, t_2], \ldots, (t_{N-1}, T]$, then the solution $s(t, x)$ in each of the intervals $[t_{k-1}, t_k]$ can be obtained successively for $k = N, N-1, \ldots, 1$ using formulas (5.17),(5.18), where T and α are replaced by t_k and α_k, respectively.

The solution in this case can be expressed in a form similar to (5.17),

$$s(t, x) = \int_0^\infty g_{N-k}(t, x; T, y)\, s(T, y) \quad (t_{k-1} \leqslant t \leqslant t_k) \tag{5.21}$$

by setting

$$g_{N-k}(t, x; T, y) = \int_0^\infty g_0(t, x; t_k, z)\, g_{N-k-1}(t_k, z; T, y)\,dz \tag{5.22}$$

The function $s(t, x)$ defined by Eqs. (5.21),(5.22) and its derivatives with respect to x of up to the second order, inclusively, are continuous for all x and t from the interval $0 \leqslant t \leqslant$ " The derivative $s_t(t, x)$ for $t = t_k$ $(k = 1, \ldots, N)$ has discontinuities of the first kind (jumps). Formulas (5.3),(5.21),(5.22) must therefore be considered as the generalized solution [4] of Eqs. (1.7),(1.8).

Example. Let us suppose that there are no measurement errors, that the intensity of the white noise ξ in Eq. (1.6) is constant and equal to R, that the control is one-dimensional, that the restriction $l(t)$ is constant and equal to l, and that the function $\omega(x)$ is defined by the equations

$$\omega(x)=0 \quad (|x|\leqslant d_0), \qquad \omega(x)=1 \qquad (|x|>d_0)$$

In this case $\Psi(x)=\omega(x)$, $R(t)=R$, $l(t)=l$ in Eqs. (1.7), (1.8). Thus, $\alpha(t)=l/R=$ $=\text{const}$ and the solution is given by formulas (5.3), (5.17), (5.20), (5.10), (5.11). As a result we obtain

$$S(t,x)=\int_{d_0}^{\infty} g_0(t,x;T,y)\,dy=\int_{d_0}^{\infty}\frac{\partial}{\partial y}\int_{\infty}^{x}[p(t,T,z-y)-e^{-2\alpha y}p(t,T,z+y)]\,dz\,dy=$$

$$=\Phi\left(\frac{x-d_0-l(T-t)}{\sqrt{R(T-t)}}\right)+\frac{1}{2}-e^{-2\frac{l}{R}d_0}\left[\Phi\left(\frac{x+d_0-l(T-t)}{\sqrt{R(T-t)}}\right)-\frac{1}{2}\right]\ (x>0) \qquad (5.23)$$

In accordance with Eqs. (2.6), approximate solution (3.1)–(3.3) in this case becomes

$$F_1(t,x)=F_2(t,x)=1-\int_{-d_0-l(T-t)}^{d_0+l(T-t)} p(t,T,x-y)\,dy=$$

$$=1-\Phi\left(\frac{x+d_0+l(T-t)}{\sqrt{R(T-t)}}\right)+\Phi\left(\frac{x-d_0-l(T-t)}{\sqrt{R(T-t)}}\right) \qquad (5.24)$$

Comparing exact solution (5.23) and approximate solution (5.24), we see that the approximate solution is in fact a lower estimate of the exact solution and that the difference between them tends to zero for $l\to 0$ or $l\to\infty$, and also in the case $R\to 0$ or $R\to\infty$.

BIBLIOGRAPHY

1. Boguslavskii, I. A., On statistically optimal control of the final state. Avtomatika i Telemekhanika №5, 1966.
2. Fel'dbaum, A. A., Principles of the Theory of Optimal Automatic Systems, 2nd ed., pp. 248-254, Moscow, "Nauka", 1966.
3. Fleming, W. H., Some Markovian optimization problems. J. Math. and Mech. Vol. 12, №1, 1963.
4. Babich, V. M., Kapilevich, M. B., Mikhlin, S. G., Natanson, G. I., Riz, P. M., Slobodetskii, L. N. and Smirnov, M. M., The Linear Equations of Mathematical Physics. (p. 18), Moscow, "Nauka", 1964.
5. Il'in, A. M., Kalashnikov, A. S. and Oleinik, O. A., Linear second-order parabolic-type equations. Uspekhi Mat. Nauk Vol. 17, №3, 1962.
6. Eidel'man, S. D., Parabolic Systems, pp 15-16, Moscow, "Nauka", 1964.
7. Krasovskii, N. N., Theory of Motion Control, p. 138, Moscow, "Nauka", 1968.
8. Berezin, I. S. and Zhidkov, N. P., Calculation Methods, Vol. 2, pp. 490-497, Moscow, Fizmatgiz, 1960.
9. Zabreiko, S. G., Rakovshchik, L. S. and Stetsenko, V. Ia., Integral Equations, pp. 154-158, Moscow, "Nauka", 1968.

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